Isoperimetric Inequalities and Applications

Inegalităţi Izoperimetrice şi Aplicaţii

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Abstract

This paper presents in the beginning the existence of the optimal form for the isoperimetric inequality. Based on the existence of the optimal form, two simple, elementary proofs are given. Some simple applications of the isoperimetric inequality are presented

Keywords: isoperimetric, optimal, inequality

1 Introduction

The isoperimetric problem is one of the simplest shape optimization problems. It is known from the antiquity that the largest area that can be enclosed by a fixed length wire should be the one of the circle, but rigorous proofs were only given in the 19th century. It can be rephrased in different equivalent ways:

- Find the greatest area which can be enclosed by a wire of length L, and the shape that realizes this maximum;
- Find the shortest length of a wire that can enclose a fixed area A, and the shape that realizes this minimum;
- For a plane figuren F with smooth boundary, if we denote its area by A and its perimeter by L we have the following inequality:

$$4\pi A \leq L^2.$$

There are numerous ways to prove this inequality:

- four hinge proof;
- Hurwitz's pfoof using Wirtinger's inequality;
- Brunn Minkowski's Inequality;
- Hadwiger's proof using Steiner's inequality;
- parametrization proof;

Some of these proofs can only be used in the two dimensional case, and some of them assume the existence of the optimal form apriori. I will present an existence argument for the general N-dimensional case, and then prove in a very simple way the isoperimetric inequality in the two dimensional case. A few applications are presented in the end.

2 Isoperimetric Inequalities

2.1 Existence of the optimal form

In what follows, we denote by $|\Omega|$ the Lebesgue measure of the measurable set Ω.

We begin by defining the perimeter of a measurable set in $\mathbb{R}^N$. For an open set $D \subset \mathbb{R}^N$, we denote $\mathcal{D}(D;\mathbb{R}^N)$ the space of functions defined on D with values in $\mathbb{R}^N$ which are C^∞ with compact support. This means that $\phi = (\phi_1, ..., \phi_N) \in \mathcal{D}(D;\mathbb{R}^N)$ if each ϕ_i is a real function with compact support in D. We endow $\mathcal{D}(D;\mathbb{R}^N)$ with the norm

$$\|\phi\|_\infty := \sup_{x\in D} \left[\left(\sum_{i=1}^{N} \phi_i(x)^2 \right) \right]^{1/2} = \sup_{x\in D} |\phi(x)|.$$

Now we can define the perimeter and relative perimeter as follows:

Definition 1. Let Ω be a measurable set in D. We call the perimeter of Ω relative to D (or simply the perimeter if $D = \mathbb{R}^N$) the number

$$P_D(\Omega) = \sup\{\int_\Omega \operatorname{div}(\phi) dx;\ \phi \in \mathcal{D}(D;\mathbb{R}^N),\ \|\phi\|_\infty \leq 1\}.$$

If $D = \mathbb{R}^N$, we denote $P_{\mathbb{R}^N}(\Omega) = P(\Omega)$.

This definition extends the case when Ω has C^1 boundary, in fact, we have the following remark.

Remark 1. If Ω is a bounded open set with C^1 boundary, then $P_D(\Omega) = \int_{\partial\Omega\cap D} d\sigma$ (where $d\sigma$ is the surface element on $\partial\Omega$).

The proof of this is not trivial, and can be found in [2].

We want to obtain some compacity result for the perimeter, which would enable us to prove the existence theorem. For this, we express $\int_\Omega \operatorname{div}(\phi)dx$ in the following manner:

$$\int_\Omega \operatorname{div}(\phi)dx = \int_D \chi_\Omega \left(\sum_{i=1}^{N} \frac{\partial \phi_i}{\partial x_i} \right) dx = \langle \chi_\Omega, \sum_{i=1}^{N} \frac{\partial \phi_i}{\partial x_i} \rangle_{\mathcal{D}'\times\mathcal{D}}$$
$$= -\sum_{i=1}^{N} \langle \frac{\partial \chi_\Omega}{\partial x_i}, \phi_i \rangle_{\mathcal{D}'\times\mathcal{D}} = -\langle \nabla\chi_\Omega, \phi\rangle_{\mathcal{D}'\times\mathcal{D}},$$

where $\mathcal{D}' = \mathcal{D}'(D;\mathbb{R}^N)$ is the space of distributions $(T_1, ..., T_N)$ with $T_i \in \mathcal{D}'(D)$. This enables us to give another definition of the perimeter in terms of Radon measures, recalling that the 1-norm is equal, using duality, to

$$\|f\|_1 = \sup\{\int_D f(x)\cdot\phi(x)dx; \phi\in\mathcal{D}(D;\mathbb{R}^N),\ \|\phi\|_\infty \le 1\}.$$

Proposition 1. Let $\mu = (\mu_1, ..., \mu_N) \in \mathcal{D}'(D;\mathbb{R}^N)$. Then μ is a Radon measure of finite total mass on $\mathbb{R}^N$ if and only if

$$\|\mu\|_1 = \sup\{\langle\mu,\phi\rangle_{\mathcal{D}'\times\mathcal{D}};\ \phi\in\mathcal{D}(D;\mathbb{R}^N),\ \|\phi\|_\infty\le 1\} < \infty.$$

For a proof see [2], Prop 2.3.4.

As a consequence of the above, we have that goven $\Omega\subset D$ measurable

$$P_D(\Omega) < \infty \Leftrightarrow \nabla\chi_\Omega \text{ is a measure of finite total mass,}$$

and

$$P_D(\Omega) = \|\nabla\chi_\Omega\|_1.$$

Proposition 2. If Ω_n and Ω are measurable subsets of $\mathbb{R}^N$ then:

(i) If $\chi_{\Omega_n}\to\chi_\Omega$ in $L^1_{loc}(D)$ then $\begin{array}{rcl} |\Omega| & \le & \liminf|\Omega_n| \\ P_D(\Omega) & \le & \liminf P_D(\Omega_n)\end{array}$

(ii) If $\chi_{\Omega_n}\to\chi_\Omega$ in $L^1(D)$ then $\begin{array}{rcl} |\Omega| & = & \liminf|\Omega_n| \\ P_D(\Omega) & \le & \liminf P_D(\Omega_n)\end{array}$

Proof. For the volume part, in (i) apply Fatou's lemma, and in (ii) it is obvious.

For the perimeter, if $\phi\in\mathcal{D}(D;\mathbb{R}^N)$, $\|\phi\|_\infty\le 1$, we have

$$\int_\Omega \text{div}(\phi)dx = \lim_n \int_{\Omega_n}\text{div}(\phi)dx \le$$
$$\liminf_n\left(\sup\{\int_{\Omega_n}\text{div}(\phi)dx; \|\phi\|_\infty\le 1\}\right) = \liminf_n P_D(\Omega_n).$$

To obtain some compacity results, we identify $\mathcal{M}_b(D)$, the space of Radon measures with the dual of $\mathcal{C}_0(D)$, the space of continuous function which tend to zero on the boundary of D, each measure defining a functional $I_\mu:\mathcal{C}_0\to\mathbb{R}$ by $I_\mu(f) = \int_D f d\mu$. The converse is also true by the Riesz representation theorem. For details see [3], Theorem 6.19. Since the dual of $\mathcal{C}_0(D)$ is separable, we have

Proposition 3. If μ_n is a sequenc of Radon measures on D for which $\|\mu_n\|_1\le C$, then there exists a subsequence μ_{n_k} and $\mu\in\mathcal{M}_b(D)$ for which $\mu_{n_k}\overset{*}{\rightharpoonup}\mu$ for the weak-$*$ topology $\sigma(\mathcal{M}_b(D), C_0(D))$.

Proof. We know by the Banach-Alaoglu theorem that the unit ball is compact in the weak-$*$ topology. Moreover, since the space is separable, the unit ball in $\mathcal{M}_b(D)$ is metrizable in the weak-$*$ topology. Therefore compactness is equivalent to sequential compactness, and therefore the conclusion follows.

For more details see [1], theorems 3.16,3.28. As a non-trivial consequence we have

Theorem 1. If Ω_n is a sequence of measurable subsets of an open set $D \subset \mathbb{R}^N$, for which we have

$$|\Omega_n| + P_D(\Omega_n) \le C, \ \forall n,$$

then there exists $\Omega \subset D$ measurable and a subsequence Ω_{n_k} for which

$$\chi_{\Omega_{n_k}} \to \chi_\Omega \text{ in } L^1_{loc}(D)$$

and

$$\nabla \chi_{\Omega_k} \overset{*}{\rightharpoonup} \nabla \chi_\Omega \text{ in } \sigma(\mathcal{M}_b(D)^N, \mathcal{C}_0(D)^N).$$

Moreover, if D has finite measure, then $\chi_{\Omega_{n_k}} \to \chi_\Omega$ in $L^1(D)$.

Proof. See [2] Theorem 2.3.10.

We finally arrived to the point where we can prove the existence of the optimal form in the isoperimetric problem.

Let D be an open set in $\mathbb{R}^N$ and $V_0 \in (0, |D|)$. Consider the following problems

(A) $P(\Omega^*) = \min\{P(\Omega) : \ \Omega \in D \text{ measurable}, |\Omega| = V_0\}$

(B) $P_D(\Omega^*) = \min\{P_D(\Omega) : \ \Omega \in D \text{ measurable}, |\Omega| = V_0\}$.

Theorem 2. Suppose that $|D| < \infty$ and $V_0 \in (0, |D|)$. Then the problems $(A), (B)$ have at least one solution.

Proof. First, check that there exists at least one measurable set $\Omega \subset D$ with $|\Omega| = V_0$. To do this, we see that we can write D as a countable union of balls, $D = \bigcup_p B_p$. For k large enough we will see that the measure of $\omega_k = \bigcup_{p \le k} B_p$ has measure greater than V_0. By continuity, there exists $r \in (0, \infty)$ such that $|\omega_k \cap B(0, r)| = V_0$, and this open set has finite perimeter, because it is obtained from a finite number of balls.

Consider now a minimising sequence (Ω_n) for problem (A). Because $P(\Omega_n) = \|\nabla \chi_{\Omega_n}\|_1$ is bounded, by Theorem 1, we can find $\Omega^* \subset D$ measurable and a subsequence $\chi_{\Omega_{n_k}}$ which converges in $L^1(D)$ to χ_{Ω^*}. From Proposition 2 we have that $|\Omega^*| = V_0$ and by lower semicontinuity of the preimeter $P(\Omega^*) \le \lim P(\Omega_n)$. This means that Ω^* is an optimal form for the problem (A).

For problem (B) we have the same reasoning, with $P(\Omega_n)$ replaced by $P_D(\Omega_n)$. With the same type of reasoning, and the use of Theorem 1 we can prove that the following dual problems have solutions:

(C) $|\Omega^*| = \max\{|\Omega| : \Omega \subset D, \ P(\Omega) = P_0\}$

(D) $|\Omega^*| = \max\{|\Omega| : \Omega \subset D, \ P_D(\Omega) = P_0\}$

As a consequence of these facts, there is an optimal figure for the isoperimetric inequality.

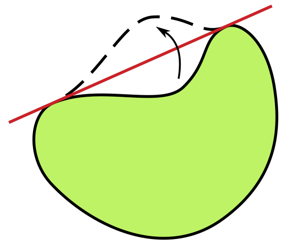

Figure 1: Optimal figure is convex

2.2 A few elementary proofs of the isoperimetric inequality

At first, it is clear that the optimal figure is convex, elseway the area can be increased, preserving perimeter, as in the figure below.

Brahmagupta's formula. It is known that the area S of the quadrilateral $ABCD$ of sides a, b, c, d satisfies the formula

$$16S^2 = (a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)-16abcd\cos^2\left(\frac{A+C}{2}\right).$$

We have the following, well known

Lemma Given $a, b, c, d > 0$ such that

$$a < b+c+d,\ b < c+d+a,\ c < a+b+d,\ d < a+b+c,$$

then

(i) there exists a quadrilateral with sides a, b, c, d.

(ii) there exists a cyclic quadrilateral with sides a, b, c, d.

Proof. (i) Suppose $a \leq b \leq c \leq d$, and denote $m = a + b$. If $m = c + d$ then $a = b = c = d$ and the quadrilateral we are seeking can be chosen to be a square. Else we have $m < c+d, c < m+d, d < m+c$, and therefore, there exists a triangle with sides m, c, d. Therefore, we have found a quadrilateral $ABCD$ for which the smallest sides are aligned. Consider the vertices as hinges and we can move the middle point of the three collinear points such that there is a quadrilateral with sides a, b, c, d.

(ii) Again, a constructive proof can be given (see [6]), but there is a quicker one using continuity. Use point (i) for the existence of a quadrilateral of sides $a = AB, b = BC, c = CD, d = DA$. Suppose that $a < b < c < d$ (if two sides are equal then some trapezoid satisfies our conditions).

Consider the following extremal conditions: (1) sides a, b are aligned; (2) sides b, c are aligned if $b+c < a+d$ or sides a, d are aligned when $a+d \leq b+c$. In the first case $\angle B+\angle D > \pi$, and in the second case $\angle A+\angle C > \pi \Leftrightarrow \angle B+\angle D < \pi$. Therefore, by continuity there exists a position where $\angle B + \angle D = \pi$, which is the desired cyclic quadrilateral.

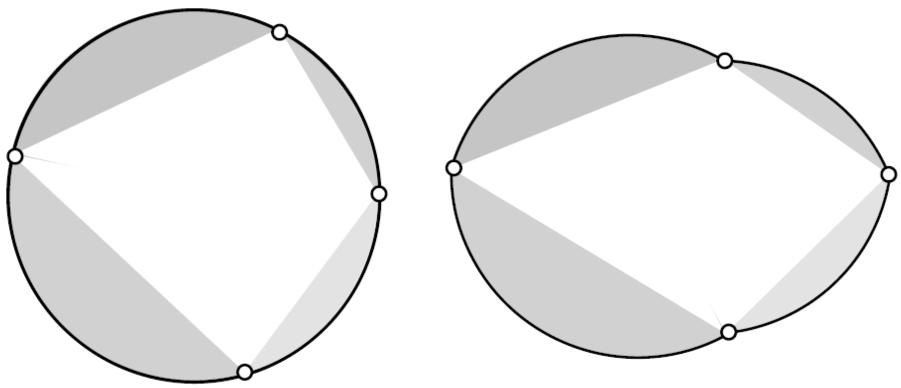

Figure 2:

Now returning to the isoperimetric inequality, suppose that the optimal figure is convex, but not a circle. Then its boundary contains four points which are not on the same circle A, B, C, D (in this order). Draw the quadrilateral $ABCD$ and consider the regions outside the quadrilateral rigid and attached to the corresponding side as in the figure. The area of $ABCD$ can be increased, preserving perimeter, by forming one cyclic quadrilateral with sides AB, BC, CD, DA like in the second part of the lemma. By Brahmagupta's formula, the neq figure has an area greater than the initial one, which was optimal. Contradiction.

Right angles. Suppose our optimal figure is convex and not a circle. First, we can find a chord AB such that AB divides the area and the perimeter in half. Indeed, it is known that given any direction d_θ, $\theta \in [0, \pi]$, there exists a unique line l_θ parallel to that direction which cuts the figure into two regions of equal areas. It is clear that $l_0 = l_\pi$, and we can consider the continuous function $f : [0, \pi] \to \mathbb{R}$, $f(\theta) = left(l_\theta) - right(l_\theta)$, which yields $f(0) + f(\pi) = 0$. By continuity, this function has at least one zero, which means that we have found a chord AB which cuts the area and perimeter in half.

Pick now a point C on one of the boundary arcs determined by AB, which can be chosen such that $\angle ACB \neq \pi/2$. Again draw triangle ACB, consider the regions outside the triangle as attached to the sides and the vertices as hinges. Then it is known that making $\angle ACB = \pi/2$ increases area of half of the figure, preserving perimeter. Taking symmetry over AB we get a new figure with the same perimeter as the initial figure, but with greater area. Contradiction. This means that the optimal figure is indeed a circle.

3 Some Applications

1. Maximal area n-gon must be cyclic. An analogue of Lemma 1 can be proved for arbitrary n-gons. The following conditions are necessary and suficient for the existence of a n-gon (even a cyclic one) with side lengths $x_i,\ i = 1..n$:

$$\sum_{i=1}^{n} x_i > 2x_j,\ j = 1..n.$$

Consider an arbitrary n-gon P constructed with sides x_i, and consider the cyclic n-gon C with the same sides x_i. Draw the circumcircle of C, and consider the regions outside the n-gon attached to the sides, and labeled $R_i,\ i = 1..n$. Attach these regions R_i to the corresponding sides of P, and obtain a figure with the same perimeter as the length of the circumcircle of C. By the isoperimetric inequality we have

$$Area(P) + \sum_{i=1}^{n} Area(R_i) \leq Area(C) + \sum_{i=1}^{n} Area(R_i),$$

which simply means that $Area(P) \leq Area(C)$. Therefore, the cyclic n-gon has the largest area.

2. Romanian TST 2005 Suppose we have a polygon with the property that the distance between any two vertices is smaller than one. Prove that the area of the polygon is smaller than $\pi/4$, and this is the best possible constant.

In the contest, the constant given was $\sqrt{3}/2$ which allowed a simpler approach.

For an arbitrary line l with polar angle $\theta \in [0, \pi]$, we denote by $r(\theta)$ the length of the projection of the polygon on the line l. There is a well known formula for the perimeter, $P = \int_0^{\pi} r(\theta) d\theta$. Since $r(\theta) \leq 1,\ \forall \theta \in [0, \pi]$, we get that $P \leq \pi$, and by the isoperimetric inequality we get that the area of our polygon is at most $\pi/4$.

3. Shortest curve which divides an equilateral triangle in two regions of equal areas.

One would try to guess this curve as being a straight line, but the answer is more trickier. First note that the curve must be simple. Suppose without loss of generality that the area of the triangle is equal to 1. There are three possible situations: (1) the curve has both its endpoints on the same side of the triangle; (2) the curve has its endpoints on two different sides of the triangle; (3) the curve is closed and inside the triangle.

Start with (3) which is the most immediate. By isoperimetric inequality, the shortest curve which encloses 1/2. This means that $L^2 \geq 2\pi$, and the shortest curve is in this case of length $L = \sqrt{2\pi}$.

Case (1). Denote XYZ our equilateral triangle and suppose that A, B, the endpoints of our curve lie on XY. Take the symmetric of the triangle XYZ about the line XY and get a triangle XYZ'. The two symmetric arcs which have endpoints in A, B enclose an area of 1 (two halves), and therefore, by the isoperimetric inequality $(2L)^2 \geq 4\pi$, which means that the shortest curve in this case has length $\sqrt{\pi}$.

Case (2). Denote again by XYZ the triangle and suppose that the endpoints A, B are on two different sides. In this case one symmetrization isn't enough.

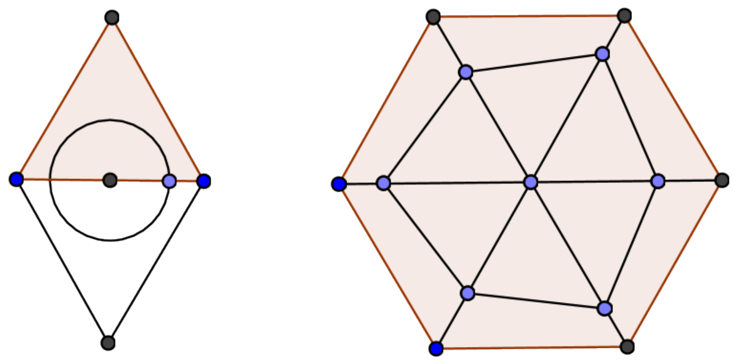

Figure 3:

We take symmetries in order to get a regular hexagon as in the figure. Then the area enclosed by the 6 copies of our curve is $6/2 = 3$, and by the isoperimetric inequality, $(6L)^2 \geq 12\pi$, which means $L^2 \geq \pi/3$ and finally the minimum length is equal to $L = \sqrt{\pi/3}$.

The last one is obviously the smallest length.

4 Conclusions

This article caught just a glimpse into the domain of isoperimetric inequalities and shape optimization. This domain is in continuous expansion and has many practical applications. I hope that the presented facts were interesting and the reader will be motivated to learn more about the facts presented.

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