INTRODUCTION

scalar cons. law.

$$\partial_t u + \operatorname{div} A(u) = 0 \quad \text{in } D'((0,\infty)\times\mathbb{R}^d)$$

family of entropy inequalities

(Ki) $\partial_t S(u) + \operatorname{div} \eta(u) \le 0$ in $D'((0,\infty)\times\mathbb{R}^d)$

for all Lipschitz cont. convex S and

$$\eta_i(\xi) = \int_0^\xi a_i(v) S'(v)\,dv$$

$$a(\cdot) = A'(\cdot) \in L^1_{loc}(\mathbb{R},\mathbb{R}^d)$$

any pair of entropy solutions u, v also satisfies

$$\partial_t |u-v| + \operatorname{div}\left[(A(u)-A(v))\operatorname{sgn}(u-v)\right] \le 0 \quad \text{in } D'((0,\infty)\times\mathbb{R}^d)$$

We can replace the entropy ineq. by a single equation named kinetic formulation.

$$\partial_t \chi(\xi,u) + a(\xi)\cdot\nabla_x \chi(\xi,u) = \partial_\xi m(t,x,\xi)$$

(KF) in $D'((0,\infty)\times\mathbb{R}^d\times\mathbb{R})$

m - entropy defect measure

$$\chi(\xi,u) = \begin{cases} 1 & 0 \le \xi \le u \\ -1 & u \le \xi \le 0 \\ 0 & \text{otherwise} \end{cases}$$

(Ki) $\Rightarrow$ (KF) use (Ki) with Kruzkov entropy and differentiate by ξ

(KF) $\Rightarrow$ (KT) multiply (KF) by $S'(\xi)$ and integrate by ξ

$$\partial_t S(u) + \operatorname{div} \eta(u) = -\int S''(\xi)\, m(t,x,\xi)\,d\xi$$

(EF)

The RHS of Ki is - a measure because every positive distribution is a measure.

Conditions on A:

LIONS, PERTHAME, TADMOR : $A_i \in C^{2,\alpha}$

EVANS PDE : A smooth

BOUCHUT PERTHAME : A Lipschitz

CONTRACTION THEOREM

$$|A|(u) = \int_0^u |a(\xi)|\,d\xi$$

Th 2.1 $u, v \in L^1_{loc}((0,\infty)\times\mathbb{R}^d)$ two entropy solutions such that $|A|(u), |A|(v) \in L^1_{loc}((0,\infty)\times\mathbb{R}^d)$. Then the contraction property holds.

If $\varphi_\varepsilon(t,x)$ is a regularizing kernel then the entropy defect measure m satisfies

$$\text{(L)} \quad \int (m(\cdot,\cdot,\xi) * \varphi_\varepsilon)(\delta(\xi - u(\cdot,\cdot)) * \varphi_\varepsilon)\,d\xi \to 0 \quad \text{in } D'((0,\infty)\times\mathbb{R}^d)$$

If q is the entropy defect measure associated to v then

$$\partial_t |u-v| + \operatorname{div}\left[(A(u)-A(v))\operatorname{sgn}(u-v)\right] = -2 \lim_{\varepsilon\to 0} \int \left[(m(\cdot,\cdot,\xi) * \varphi_\varepsilon)(\delta(\xi-v) * \varphi_\varepsilon) + (q(\cdot,\cdot,\xi) * \varphi_\varepsilon)(\delta(\xi-u) * \varphi_\varepsilon)\right] d\xi$$

[δ is the Dirac mass]

PROOF: STEP I - Continuity

$\xi \mapsto m(x,t,\xi)$ is cont in $D'((0,\infty)\times\mathbb{R}^d)$

and $\partial_t |u-\xi_0| + \operatorname{div} \eta_{\xi_0}(u) = -2m(t,x,\xi_0)$ in $D'((0,\infty)\times\mathbb{R}^d)$

η_ξ is the entropy flux associated to Kruzkov's entropy.

Pick $S_\delta(u) \to |u-\xi_0| - |\xi_0|$ in (EF)

Proof of (L) $\leftarrow$ STEP II consider $\varphi_\varepsilon(t,x)$ a regularizing kernel.

$$\left[(KF) * \varphi_\varepsilon(t,x)\right] \cdot 2\chi(\xi,u) * \varphi_\varepsilon$$

then integrate with respect to ξ

PROOF THAT (CP) + continuity of solution in $t=0$ implies uniqueness: (Cf. Evans PDE, pag 610)

$$\partial_t |u-v| + \operatorname{div}\,(A(u)-A(v))\operatorname{sgn}(u-v) \le 0 \quad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}^d)$$

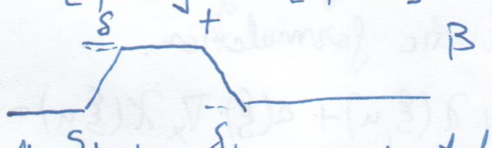
$\Leftrightarrow$ $\forall\, \varphi \in C_c^\infty((0,\infty)\times\mathbb{R}^d)$, $\varphi \ge 0$ we have

$$(CP) \quad \int_0^\infty \int_{\mathbb{R}^d} \left(|u-v|\,\partial_t\varphi + [A(u)-A(v)]\operatorname{sgn}(u-v)\,\partial_x\varphi\right) dx\,dt \gtrless 0$$

(int. by parts gives a − sign and changes the inequality ?!?)

Pick $0<s<t$, $r>0$, $\varphi(x,t) = \alpha(x)\beta(t)$

$$\begin{cases} \alpha : \mathbb{R}^d \to \mathbb{R}^d \text{ smooth} \\ \alpha(x) = 1 \text{ for } |x| \le r, \ \cancel{\alpha(x)=0} \\ \alpha(x) = 0 \text{ for } |x| \ge r+1 \\ |\alpha'(x)| \le 2 \text{ for } |x| \in (r, r+1) \end{cases} \qquad \begin{cases} \beta : \mathbb{R}\to\mathbb{R} \ ([0,\infty)\to[0,\infty)) \text{ Lipschitz} \\ \beta(\tau) = 0 \quad 0 \le \tau \le s, \ \tau \ge t+\delta \\ \beta(\tau) = 1 \quad s+\delta \le \tau \le t \\ \beta \text{ linear on } [s, s+\delta] \text{ and } [t, t+\delta] \end{cases}$$



β is not smooth, but can be approximated by smooth functions.

Replace φ in (CP) and obtain

$$\frac{1}{\delta}\int_s^{s+\delta}\int_{\mathbb{R}^d} |u-v|\,\alpha(x)\,dx\,d\tau \ge \frac{1}{\delta}\int_t^{t+\delta}\int_{\mathbb{R}^d} |u-v|\,\alpha(x)\,dx\,d\tau - \int_s^{t+\delta}\int_{r\le|x|\le r+1} [A(u)-A(v)]\operatorname{sgn}(u-v)\,\alpha'(x)\beta(\tau)\,dx\,d\tau$$

Take $r\to\infty$ and get

$$\frac{1}{\delta}\int_s^{s+\delta}\int_{\mathbb{R}^d} |u-v|\,dx\,d\tau \ge \frac{1}{\delta}\int_t^{t+\delta}\int_{\mathbb{R}^d} |u-v|\,dx\,d\tau$$

Take $\delta\to 0$ and get

$$\|u(s,\cdot)-v(s,\cdot)\|_{L^1(\mathbb{R}^d)} \ge \|u(t,\cdot)-v(t,\cdot)\|_{L^1(\mathbb{R}^d)} \quad \forall\, 0<s<t.$$

Take $s\to 0$ and if $u^0 = v^0$ we see that $u = v$ a.e. $\Rightarrow$

$\Rightarrow$ (SCL) + (EI) yields a unique solution.

and get

$$\partial_t \int (\chi(\xi,u)*\varphi_\varepsilon)^2 d\xi + \operatorname{div} \int a(\xi)(\chi(\xi,u)*\varphi_\varepsilon)^2 d\xi = 2\int \chi(\xi,u)*\varphi_\varepsilon \cdot \partial_\xi m(t,x,\xi)*\varphi_\varepsilon\, d\xi$$

$$= -2\int \partial_\xi \chi(\xi,u)*\varphi_\varepsilon \cdot m(t,x,\xi)*\varphi_\varepsilon \, d\xi$$

$$= -2\int m(t,x,\xi)*\varphi_\varepsilon \,[\delta(\xi)-\delta(\xi-u(t,x))]*\varphi_\varepsilon \, d\xi$$

For $\varepsilon \to 0$ we have

$$\int (\chi(\xi,u)*\varphi_\varepsilon)^2 d\xi \to |u| \quad L^1_{loc}$$

$$\int a(\xi)(\chi(\xi,u)*\varphi_\varepsilon)^2 d\xi \to \eta_0 \quad L^1_{loc}$$

$$\int m(t,x,\xi)*\varphi_\varepsilon\, \delta(\xi)*\varphi_\varepsilon\, d\xi = m(t,x,0)*\varphi_\varepsilon \to m(t,x,0) \text{ in } D'$$

Hence the last term also converges to zero, using step I with $\xi_0 = 0$

STEP III – contraction

Cons u, v as in the statement of Th 2.1 and m, q the entropy defect measures.

Consider a nonnegative regularising kernel φ_ε

$[(KF)_u - (KF)_v] \cdot 2(\chi(\xi,u)-\chi(\xi,v))*\varphi_\varepsilon$, and integrate with respect to ξ

$$\partial_t \int [(\chi(\xi,u)-\chi(\xi,v))*\varphi_\varepsilon]^2 d\xi + \operatorname{div}\int a(\xi)[(\chi(\xi,u)-\chi(\xi,v))*\varphi_\varepsilon]^2 d\xi =$$

$$= -2\int (m(t,x,\xi)-q(t,x,\xi))*\varphi_\varepsilon\, \partial_\xi(\chi(\xi,u)-\chi(\xi,v))*\varphi_\varepsilon\, d\xi$$

$$= -2\int (m(t,x,\xi)-q(t,x,\xi))*\varphi_\varepsilon\, [\delta(\xi-v(t,x))-\delta(\xi-u(t,x))]*\varphi_\varepsilon\, d\xi$$

only terms with –

$$\le 2\int [(m(t,x,\xi)*\varphi_\varepsilon)(\delta(\xi-u(t,x))*\varphi_\varepsilon) + (q(t,x,\xi)*\varphi_\varepsilon)(\delta(\xi-v(t,x))*\varphi_\varepsilon)]\, d\xi \xrightarrow[\varepsilon\to 0]{} 0$$

Take $\varepsilon \to 0$ and use

$$\int [(\chi(\xi,u)-\chi(\xi,v))*\varphi_\varepsilon]^2 d\xi \to |u-v| \quad \text{in } L^1_{loc}$$

$$\int a(\xi)[(\chi(\xi,u)-\chi(\xi,v))*\varphi_\varepsilon]^2 d\xi \to (A(u)-A(v))\operatorname{sgn}(u-v) \quad \text{in } L^1_{loc}$$

$\Rightarrow$ contraction inequality.

STEP IV the ~~limit~~ ~~part~~ of the theorem was just proved passing to the limit in the last inequality

PROP 2.2

Let $u \in L^\infty((0,\infty) \times L^1(\mathbb{R}^d))$ be an entropy solution of (1.1), (1.2) which is cont. at $t=0$ and satisfies $|A|(u) \in L^\infty((0,\infty); L^1(\mathbb{R}^d))$. Then for any regularizing kernels $\varphi_\varepsilon(x)$, $\varphi_\alpha(t)$ the quantity

$$M_{\varepsilon,\alpha}(t,x) = \int m(\cdot,\cdot,\xi) *_x \varphi_\varepsilon *_t \varphi_\alpha \, \delta(\xi - u(\cdot,\cdot)) *_x \varphi_\varepsilon *_t \varphi_\alpha \, d\xi$$

satisfies

$$\int_0^\infty \Big[\lim_{\alpha\to 0} \int_{\mathbb{R}^d} M_{\varepsilon,\alpha}(t,x)\,dx\Big] dt \le \int_{\mathbb{R}^d} \Big(|u^0| - \int_\xi (\chi(\xi,u^0) * \varphi_\varepsilon)^2 d\xi \Big) dx \le C\varepsilon \|u^0\|_{BV}$$

<u>PROOF</u>: Similar to Thm 1.

$$\partial_t \int (\chi(\xi,u) * \varphi_\varepsilon * \varphi_\alpha)^2 d\xi + \operatorname{div} \int a(\xi) (\chi(\xi,u) * \varphi_\varepsilon * \varphi_\alpha)^2 d\xi = \cancel{-2m(t,x,0)}$$

$$= -2m(t,x,0) * \varphi_\varepsilon * \varphi_\alpha + M_{\varepsilon\alpha}(t,x)$$

Integrate in x assuming $|A|(u) \in L^1_x$ so that the div vanishes

$$\partial_t \int_{\mathbb{R}^d} \int (\chi(\xi,u) * \varphi_\varepsilon * \varphi_\alpha)^2 d\xi\, dx = -2 \int_{\mathbb{R}^d} m(t,x,0) * \varphi_\alpha \,dx + \int_{\mathbb{R}^d} M_{\varepsilon,\alpha}(t,x)\,dx$$

$$\underset{\int \operatorname{div} = 0}{\overset{\text{use step I Th 2.1}}{=}} \partial_t \int_{\mathbb{R}^d} |u(t,x)| * \varphi_\alpha \,dx + \int_{\mathbb{R}^d} M_{\varepsilon,\alpha}(t,x)\,dx$$

Take $\alpha \to 0$ and get

$$\partial_t \int_{\mathbb{R}^d} \Big[\int (\chi(\xi,u) * \varphi_\varepsilon)^2 d\xi - |u(t,x)| \Big] dx = \lim_{\alpha\to 0} \int_{\mathbb{R}^d} M_{\varepsilon,\alpha}(t,x)\,dx$$

Because of the continuity in t, we may integrate in t and get

$$\int_0^t \lim_{\alpha\to 0} \int_{\mathbb{R}^d} M_{\varepsilon\alpha}(s,x)\,dx\,ds = \int_{\mathbb{R}^d} \Big[\int (\chi(\xi, u(t,\cdot)) * \varphi_\varepsilon)^2 d\xi - |u(t,x)| \Big] dx -$$

$$- \int_{\mathbb{R}^d} \Big[\int (\chi(\xi,u^0) * \varphi_\varepsilon)^2 d\xi - |u^0(x)| \Big] dx$$

It remains to check that

$$\int_{\mathbb{R}^d} \int (\chi(\xi, u(t,\cdot)) * \varphi_\varepsilon)^2 \, d\xi \, dx \le \int_{\mathbb{R}^d} |u(t,x)| \, dx$$

just replace the convolution and calculate.
or notice that the reg of piecewise const function looks like this:

and

$$0 \le \int_{\mathbb{R}^d} \left[|u^0(x)| - \int (\chi(\xi, u^0) * \varphi_\varepsilon)^2 \, d\xi \right] \le C \|u^0\|_{BV} \cdot \varepsilon$$

just like above

Denote $F(u) = \int_{\mathbb{R}^d} \left[|u(t,x)| \cancel{dx} - \int_\xi (\chi(\xi, u(t,\cdot)) * \varphi_\varepsilon)^2 \right] dx.$

Then $F(u) \ge 0$

$$F(u^0) - F(u) \ge 0$$

$$F(u) \le F(u^0) \le C \|u^0\|_{BV} \, \varepsilon.$$

(estimate will be used in the proof of Th 3.1)

ERROR ESTIMATES

Consider an entropy solution u of $\begin{cases} \partial_t u + \operatorname{div} A(u) = 0 & \text{in } \mathcal{D}' \\ \partial_t S(u) + \operatorname{div} \eta(u) \le 0 & \text{in } \mathcal{D}' \end{cases}$

and v solve in $\mathcal{D}'$ the approx. equation

$$\partial_t \chi(\xi,v) + a(\xi)\cdot \nabla_x \chi(\xi,v) = \partial_\xi \left[q(t,x,\xi) + D_x^j e(t,x,\xi) \right]$$

Th 3.1

if $u, |A|(u), v, |A|(v) \in L^1_{loc}((0,\infty)\times\mathbb{R}^d)$ like above then

(i) For nonnegative kernels $\varphi_\varepsilon(x)$, $\psi_\alpha(t)$ and the defect measure $M_{\varepsilon,\alpha}$ defined in Prop 2.2. we have

$$\partial_t \int (\chi(\xi,u)*\varphi_\varepsilon*\psi_\alpha - \chi(\xi,v))^2 d\xi + \operatorname{div} \int a(\xi)(\chi(\xi,u)*\varphi_\varepsilon*\psi_\alpha - \chi(\xi,v))^2 d\xi$$

$$\le M_{\varepsilon,\alpha}(t,x) - 2\int D_x^j e(t,x,\xi)\, \delta(\xi - u(t,x))*\varphi_\varepsilon*\psi_\alpha \, d\xi$$

(ii) For $u^0 \in L^1\cap BV(\mathbb{R}^d)$, $u,v \in C^0((0,\infty): L^1(\mathbb{R}^d))$ and $|A|(u), |A|(v) \in L^\infty((0,\infty); L^1(\mathbb{R}^d))$ we have

$$\|u(T,\cdot) - v(T,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} + C\|u^0\|_{BV}^{j/j+1} \cdot |||e(\cdot,\cdot,\cdot)|||_T^{1/j+1}$$

PROOF: STEP I $\quad \partial_t |v| + \operatorname{div}(\underbrace{A(v)\operatorname{sgn}(v)}_{\eta_0(v)}) = -2\left[q(t,x,0) + D_x^j e(t,x,0)\right]$

STEP II Introduce a regularizing kernel $\varphi_\nu(t,x)$

$$\partial_t \int (\chi(\xi,v)*\varphi_\nu)^2 d\xi + \operatorname{div}\int a(\xi)(\chi(\xi,v)*\varphi_\nu)^2 d\xi =$$

$$= -2\left(q(t,x,0)*\varphi_\nu + D_x^j(t,x,0)*\varphi_\nu\right) +$$

$$+2\int (q(t,x,\xi) + D_x^j e(t,x,\xi))*\varphi_\nu \, \delta(\xi - v(t,x))*\varphi_\nu \, d\xi$$

same proof as step II from Th 2.1

STEP III Combine the kinetic formulations of u,v.

$$\partial_t \int (\chi(\xi,u)*\varphi_\varepsilon*\psi_\alpha - \chi(\xi,v)*\varphi_\nu)^2 d\xi + \operatorname{div}\int a(\xi)(\chi(\xi,u)*\varphi_\varepsilon*\psi_\alpha - \chi(\xi,u)*\varphi_\nu)^2 d\xi =$$

$$= -2\int \left(m(t,x,\xi)*\varphi_\varepsilon*\psi_\alpha - \boxed{q(t,x,\xi)*\varphi_\nu}\right)\left(\boxed{\delta(\xi - v(t,x))*\varphi_\nu} - \delta(\xi - u(t,x))*\varphi_\varepsilon*\psi_\alpha\right) d\xi$$

$$+2\int \boxed{D_x^j e(t,x,\xi)*\varphi_\nu}\left(\boxed{\delta(\xi - v(t,x))*\varphi_\nu} - \delta(\xi-u(t,x))*\varphi_\varepsilon*\psi_\alpha\right) d\xi$$

as a consequence of steps I and II, terms with green vanish as $\nu \to 0$ so we obtain exactly (i)

STEP IV Integrate (i) in x. After integrating by parts, RHS becomes:

$$\int_{\mathbb{R}^d} M_{\varepsilon,\alpha}(t,x)\,dx + 2\cdot(-1)^j \int_{\mathbb{R}^d} \int e(t,x,\xi)\,\delta(\xi-u) * (D_x^j \varphi_\varepsilon) * \psi_\alpha \, d\xi\, dx$$

$$\leq \int_{\mathbb{R}^d} M_{\varepsilon,\alpha}(t,x)\,dx + 2\int_{\mathbb{R}^d} \sup_\xi |e(t,x,\xi)| \int \delta(\xi-u) * |D_x^j \varphi_\varepsilon| * \psi_\alpha \, d\xi\, dx$$

Then

$$\int_{\mathbb{R}^d} |u-v|\,dx \leq 2\int_{\mathbb{R}^d}\int \left[(\chi(\xi,u)*\varphi_\varepsilon - \chi(\xi,u))^2 + (\chi(\xi,u)*\varphi_\varepsilon - \chi(\xi,v))^2\right] d\xi\, dx$$

$$\leq C\|u\|_{BV}\,\varepsilon + \int_{\mathbb{R}^d}\int (\chi(\xi,u)*\varphi_\varepsilon - \chi(\xi,u))^2\, d\xi\, dx$$

The first ineq is true because $(a-b)^2 \leq 2[(a-c)^2 + (b-c)^2]$

$$\int_{\mathbb{R}^d} |u-v|\,dx = \int_{\mathbb{R}^d}\int_\xi [\chi(\xi,u)-\chi(\xi,v)]^2 d\xi\, dx \leq 2\int_{\mathbb{R}^d}\int [(\chi(\xi,u)*\varphi_\varepsilon - \chi(\xi,u))^2 + (\chi(\xi,u)*\varphi_\varepsilon - \chi(\xi,v))^2]\, d\xi\, dx$$

$$\int_0^T\int_{\mathbb{R}^d} \int (\chi(\xi,u)*\varphi_\varepsilon - \chi(\xi,v))^2 d\xi\, dx \leq \int_0^T \lim_{\alpha\to 0}\int_{\mathbb{R}^d} M_{\varepsilon\alpha}(t,x)\,dx +$$

$$+ 2\int_0^T\int_{\mathbb{R}^d} \sup_\xi |e(t,x,\xi)| \int \delta(\xi-u) * |D_x^j\varphi_\varepsilon| * \psi_\alpha\, d\xi\, dx \leq$$

$$\leq C\|u^0\|_{BV}\,\varepsilon + C\varepsilon^{-j}\,|||e(\ldots)|||$$

$$\Rightarrow \int_0^T\int_{\mathbb{R}^d} |u(t,\cdot) - v(t,\cdot)|\,dx\,dt \leq 2\int_0^T F(u(t,\cdot))\,dt + C\|u^0\|_{BV}\,\varepsilon + C\varepsilon^{-j}\,|||e(\ldots)|||$$

$$\leq C\|u^0\|_{BV}\,\varepsilon + C\varepsilon^{-j}\,|||e(\ldots)|||. = G(\varepsilon)$$

After optimizing ε (find ε_0 sol of $G'(\varepsilon)=0$) we get the desired result:

$$\|u(T,\cdot) - v(T,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} + C\|u^0\|_{BV}^{j/j+1}\, |||e(\ldots)|||_T^{1/j+1}$$

Ref: LIONS, PERTHAME, TADMOR

APPLICATION

Consider a diffusion equation

$$(E) \qquad \partial_t v + \operatorname{div} A(v) = \varepsilon \Delta v$$

with a $L^1 \cap BV$ initial data. For this approximation we have

$$\| u(T,\cdot) - v(T,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|u^0 - v^0\|_{L^1(\mathbb{R}^d)} + C(\varepsilon T \|u^0\|_{BV} \|v^0\|_{BV})^{1/2}$$

The equation E, together with the entropy inequality

$$(EI) \quad \partial_t S(v) + \operatorname{div} \eta(v) - \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} \eta_{ij}(v) \le 0 \quad \text{in } D'$$

$$\eta_i(t) = \int_0^t S'(s) A_i'(s)\, ds \qquad \eta_{ij}(t) = \int_0^t S'(s) A_{ij}'(s) \qquad , (A_{ij}) = \varepsilon I : \mathbb{R}^d \to \mathbb{R}^d$$

in our case

so $\eta_{ij}(t) = 0$ for $i \ne j$ and so

and $\eta_{ii} = \varepsilon(S(t) - S(0))$

(EI) becomes

$$\partial_t S(v) + \operatorname{div} \eta(v) - \varepsilon \Delta S(v) \le 0 \quad \text{in } D'$$

The kinetic formulation for this form of entropy ineq. is

$$\partial_t \chi(\xi, v) + a(\xi) \cdot \nabla_x \chi(\xi, v) = \partial_\xi q(t,x,\xi) + \varepsilon \Delta_x \chi(\xi, v)$$

$$q = \varepsilon \delta(\xi - v) \frac{\partial \chi}{\partial x_i} \cdot \frac{\partial \chi}{\partial x_i} = \varepsilon \delta(\xi - v) |\nabla v|^2 \ge 0$$

$$\varepsilon \Delta_x \chi(\xi, v) = \partial_\xi |D_x^1 e(t,x,\xi)| \quad . \quad \text{Integrate in } x$$

$$\partial_\xi e(t,x,\xi) = \varepsilon \nabla_x \chi(\xi, v) = \varepsilon \delta(\xi - v) \nabla_x v$$

we can take $e(t,x,\xi) = \varepsilon \mathbb{1}_{\{v \le \xi\}} \nabla_x v$

$$|||e(t,x,\xi)|||_T \le T\varepsilon \int_{\mathbb{R}^d} |\nabla_x v|\, dx \le T\varepsilon \|v^0\|_{BV}$$

using Th 3.1 we get the desired estimate.